AD-A116 177   WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER   F/G 12/1
                FINITE DIFFERENCE METHODS FOR THE STOKES AND NAVIER-STOKES EQUA--ETC(U)
                APR 82   J C STRIKWERDA   DAAG29-80-C-0041
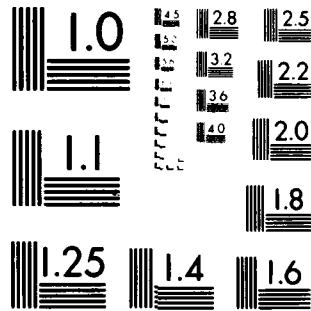UNCLASSIFIED   MRC-TSR-2361   NL

END
DATE
FILMED
7 82
DTIC

MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS 1963 A
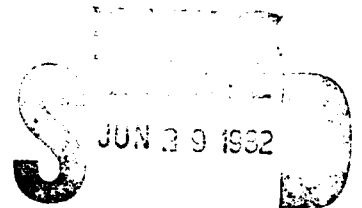
AD A116177

MRC Technical Summary Report #2361

FINITE DIFFERENCE METHODS FOR THE
STOKES AND NAVIER-STOKES EQUATIONS

John C. Strikwerda

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

April 1982

(Received March 15, 1982)

JUN 2 9 1982

A

Approved for public release
Distribution unlimited

82 06 29 033

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

FINITE DIFFERENCE METHODS FOR THE
STOKES AND NAVIER-STOKES EQUATIONS

John C. Strikwerda

Technical Summary Report #2361
April 1982

ABSTRACT

This paper presents a new finite difference scheme for the Stokes
equations and incompressible Navier-Stokes equations for low Reynold's
number. The scheme uses the primitive variable formulation of the equations
and is applicable with non-uniform grids and non-rectangular geometries.
Several other methods of solving the Navier-Stokes equations are also examined
in this paper and some of their strengths and weaknesses are described.
Computational results using the new scheme are presented for the Stokes
equations for a region with curved boundaries and for a disc with polar
coordinates. The results show the method to be second-order accurate.

AMS (MOS) Subject Classifications: 65N05, 76D05

Key Words: Finite Differences, Navier-Stokes Equations, Stokes Equations

Work Unit Number 3 (Numerical Analysis and Computer Science)

SIGNIFICANCE AND EXPLANATION

The incompressible Navier-Stokes equations describe the flow of many common fluids such as water or air at low speeds. Thus numerical methods for solving these equations are very important for many scientific and engineering applications. In this paper a new second-order accurate finite difference method for solving the incompressible Navier-Stokes equations is presented. An advantage of this new scheme over other methods is that it can be applied with non-rectangular regions or non-uniform grids.

In this paper several other methods for solving the Navier-Stokes equations are examined and some of their strengths and weaknesses are discussed.

FINITE DIFFERENCE METHODS FOR THE
STOKES AND NAVIER-STOKES EQUATIONS

John C. Strikwerda

 1. INTRODUCTION. In this paper we examine several common methods for solving the incompressible Navier-Stokes equations by finite differences and we present a new second-order accurate finite difference scheme for these equations. This new scheme is designed to be applied with non-uniform grids and non-orthogonal coordinate systems. Numerical experiments with the Stokes equations illustrate the versatility and accuracy of the scheme.

 The steady-state Stokes equations on a domain Ω in R^n are given by

$$-\nabla^2 \vec{u} + \vec{\nabla} p = \vec{f}(x)$$

(1.1)

$$\vec{\nabla} \cdot \vec{u} = g(x)$$

and the steady-state Navier-Stokes equations are

$$-R^{-1}\nabla^2 \vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{u} + \vec{\nabla} p = \vec{f}(x)$$

(1.2)

$$\vec{\nabla} \cdot \vec{u} = g(x)$$

where R is the Reynolds number. We will consider the systems (1.1) and (1.2) with Dirichlet boundary conditions

(1.3)
$$\vec{u}(x) = \vec{b}(x) \quad \text{on} \quad \partial\Omega .$$

 A necessary condition for (1.1) or (1.2) to have a solution is that the data $g(x)$ and $b(x)$ satisfy the integrability condition

(1.4)
$$\int_\Omega g = \int_{\partial\Omega} \vec{b} \cdot \vec{n} ,$$

where $\vec{n}$ is the outer unit normal to $\partial\Omega$. For the mathematical theory of the systems (1.1) and (1.2) we refer to Ladyzhenskaya (1963) and Teman (1979).

 We will be concerned only with methods that solve the systems (1.1) and (1.2) in the primitive variables u and p and not with methods such as the

---

vorticity and stream-function reformulation. Also our methods are applicable in two or three dimensions although our examples will be only in two dimensions.

We emphasize that the scheme presented here is designed to be easily applicable with non-rectangular geometries and non-uniform grids. The vast majority of papers on the numerical solution of the incompressible Navier-Stokes equations limit themselves to examples using rectangular geometry and uniform grids. By way of contrast, computations with the compressible Navier-Stokes equations routinely use non-rectangular geometries and non-uniform grids.

The outline of the remaining sections of the paper is as follows. In Section 2 we discuss the strengths and weaknesses of some common approaches to solving the systems (1.1) and (1.2) and in Section 3 we discuss finite difference schemes for these systems. The finite difference integrability condition is discussed in Section 4, and computational results are given in Section 5. The numerical examples of Section 5 demonstrate that the new scheme given here can be used to give second-order accurate solutions to the Stokes equations for non-rectangular geometries. To our knowledge no other finite difference schemes for the Stokes or incompressible Navier-Stokes equations in the primitive variables have been shown to be second-order accurate for non-rectangular geometries. Computations using the new scheme for the incompressible Navier-Stokes equations are currently being made and will be reported when complete.

2. SOLUTION TECHNIQUES. In this section we review some approaches to solving the Navier-Stokes and Stokes equations numerically. Few researchers have treated the system (1.2) in the given form, most have altered it in some way. Before examining the altered forms of (1.2) we look at the system in the given form.

The Stokes equations (1.1) and the Navier-Stokes equations (1.2) are elliptic systems of $n + 1$ equations in $n + 1$ dependent variables. The definition of an elliptic system, as given by Douglis and Nirenberg (1957), requires that the determinant of the principle symbol of the system not vanish for non-zero values of dual variables. For the Navier-Stokes equations the determinant of the principle symbol is

$$(2.1) \qquad \det \begin{pmatrix} \frac{1}{R} |\xi|^2 I_n & i\vec{\xi} \\ i\vec{\xi}^T & 0 \end{pmatrix} = |\xi|^{2n}$$

which is non-zero for $|\xi| \neq 0$. Moreover, since the determinant is a polynomial of degree $2n$ in the variables $\xi = (\xi_1,\ldots,\xi_n)$ the system requires n boundary conditions at each point of the boundary (Agmon, Douglis and Nirenberg (1964)). These boundary conditions will usually be Dirichlet or Neumann conditions on the velocity u.

One of the most common ways of modifying the Navier-Stokes equations
(1.2) is to replace it by the system

$$-R^{-1}\nabla^2\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = \vec{f}(x)$$

(2.2)

$$\nabla^2 p = \vec{\nabla}\cdot\vec{f} + R^{-1}\nabla^2 g - \sum_{i,j} u^i_{x_j} u^j_{x_i} - \vec{u}\cdot\vec{\nabla}g$$

The last equation of (2.2) is obtained by taking the divergence of the first
equation of (1.2) and then using the last equation of (1.2) to eliminate the
divergence of velocity. The system (2.2) has the advantage over (1.2) in
that, when discretized, it can be solved using standard methods for inverting
the discrete Laplacian. However, the system (2.2) has a grave disadvantage in
that it requires n + 1 boundary conditions, one for each elliptic equation,
as opposed to (1.2) which requires only n boundary conditions. Thus any
attempt to solve (1.2) via (2.2) would require some means of determining the
correct additional boundary condition. Without the correct condition
solutions of (2.2) will not be solutions of (1.2).

Roache (1972, p. 194) suggests that the additional boundary condition be
given by the normal derivative of pressure as determined by the first equation
of (1.2) or (2.2) evaluated on the boundary. This, however, is not
satisfactory as a boundary condition since it is not independent of the system
of differential equations. Roache's suggestion leaves the system (2.2)
underdetermined.

Another boundary condition which is commonly used along boundaries
corresponding to physical surfaces is to set the normal derivative of the
pressure to zero, which is valid in the limit for high Reynolds number flow.
With this boundary condition and (1.3) the system (2.2) has the proper number
of boundary conditions, however, its solutions are not solutions of (1.2).

As one would expect, the methods using (2.2) or similar systems have
difficulty with the accuracy of the pressure field and with satisfying the
incompressibilty condition on the velocities (see for example the work by
Boney, Hefner, Hirsh, and Zoby reported in Rubin and Harris (1975)).

The above mentioned difficulties are seen in computations with the time-
dependent Navier-Stokes equations as well. Roache (1972) has a discussion of
the difficulties of obtaining a zero divergence for the velocity field when
using the above approach for time dependent flows (see also Harlow and Welch
(1964)).

Because of these difficulties, it seems best not to use the derived
system (2.2) but to use the original system (1.2).

Another approach to solving the Navier-Stokes equations (1.2) is the
artificial compressibility method. The basic idea of this method is to solve
a time-dependent system of equations, whose steady-state solutions solve
(1.2), until a steady state is reached. Methods have been proposed by Chorin
(1967) and Yanenko (1967). The convergence rate of these methods is dependent
on the choice of finite difference method used to solve the system. Moreover,
as will be discussed in Section 4, it may happen that the finite difference

equations do not have a steady-state solution, so the method cannot converge. Taylor and Ndefo (1970) reported difficulty in getting Yanenko's method to converge, most likely because there was no solution.

Another common method is to use the "parabolized" Navier-Stokes equations in which the second-derivatives in the stream-wise direction are removed. Because of its limited applicability and uncertain justification we will not discuss this method here except to note that often an analogue of (2.2) is derived and thus some of our observations on (2.2) also apply to the parabolized equaitons. Raithby and Schneider (1979) discuss these difficulties for three-dimensional flow problems.

3. FINITE DIFFERENCE SCHEMES. In this section we discuss the staggered mesh and central finite difference schemes for (1.1) and (1.2) and introduce a new scheme. The second-order accurate staggered mesh scheme for a uniform cartesian grid assigns the values of each of the velocity components and the pressure to different interlaced grids. In two dimensions with velocity components u and v, one may assign values of u to grid locations $\left(\left(i + \frac{1}{2}\right)h, jh\right)$, values of v to $\left(ih, \left(j + \frac{1}{2}\right)h\right)$, and values of p to (ih, jh), e.g. Harlow and Welch (1965), Patankar and Spalding (1972), Raithby and Schneider (1979), Brandt and Dinar (1979). This method works very well as long as the geometry is rectangular and the grid is uniform. Non-uniform grids and grid mapping techniques cannot be conveniently handled.

The staggered mesh schemes also have some difficulty at boundaries. For example, when both velocity components are specified at a boundary then that velocity component whose mesh lines do not lie on the boundary requires some special treatment.

The central difference scheme on a uniform rectangular mesh assigns values of all the variables to each grid point. The divergence and gradient operators are approximated using central differences and the Laplacian is approximated by the standard five-point discrete Laplacian. Central difference schemes have been used by Chorin (1967, 1968) in time-dependent calculations.

An important concept for finite difference schemes for elliptic systems such as (1.1) and (1.2) is that of regularity (see Bube and Strikwerda (1980), and also Frank (1968), Brandt and Dinar (1979)). Regular schemes give rise to regularity estimates analogous to those in the theory of elliptic systems of differential equations. Solutions to regular difference schemes will in general be smoother than solutions to non-regular schemes and also will be more accurate approximations to the solutions of the differential equations.

The central difference scheme is non-regular (Bube and Strikwerda (1980)), which results in non-smooth solutions. The lack of smoothness is most noticeable in the pressure. The staggered mesh scheme is regular. The advantage of the central difference scheme is that it is easily implemented with non-uniform grids as introduced by coordinate changes.

It should be emphasized that none of the difficulties mentioned above are insurmountable. Both the staggered mesh and central differencing schemes have been used and often quite successfully. However we will consider a new scheme

which incorporates both regularity and ease of implementation with coordinate grid mapping techniques.

Before introducing the new scheme we will discuss the concept of regularity for difference schemes as given in Bube and Strikwerda (1980). A difference operator A may be written as

$$Af(x) = \sum_{\mu} a_{\mu}(h,x)T^{\mu}f(x) ,$$

where T^{μ} is the translation operator given by

$$T^{\mu}f(x_{\nu}) = f(x_{\nu+\mu})$$

for multi-indices ν and μ.

The symbol of A is given by

$$a(h,x,\zeta) = \sum_{\mu} a_{\mu}(h,x)e^{i\mu \cdot \zeta} .$$

For example, the first-order central difference operator in the k-th coordinate direction has symbol

$$\frac{e^{i\zeta_k} - e^{-i\zeta_k}}{2h_k} = ih_k^{-1}\sin \zeta_k$$

and the standard second-order accurate Laplacian in n-variables has the symbol

$$- \sum_{k=1}^{n} 4h_k^{-2}\sin^2 \tfrac{1}{2} \zeta_k .$$

A finite difference scheme for the Stokes equations is regular elliptic if the determinant of the matrix of symbols of the scheme vanishes only for ζ equal to zero modulo 2π. For the Stokes equations with central differencing, and $\Delta x = \Delta y = h$, this determinant is

$$\det \begin{pmatrix} 4h^{-2}(\sin^2 \tfrac{1}{2} \zeta_1 + \sin^2 \tfrac{1}{2} \zeta_2) & 0 & ih^{-1}\sin \zeta_1 \\ 0 & 4h^{-2}(\sin^2 \tfrac{1}{2} \zeta_1 + \sin^2 \tfrac{1}{2} \zeta_2) & ih^{-1}\sin \zeta_2 \\ ih^{-1}\sin \zeta_1 & ih^{-1}\sin \zeta_2 & 0 \end{pmatrix}$$

$$= 4h^{-4}(\sin^2 \tfrac{1}{2} \zeta_1 + \sin^2 \tfrac{1}{2} \zeta_2)(\sin^2\zeta_1 + \sin^2\zeta_2) .$$

This determinant vanishes for the dual variables ζ_1 and ζ_2 equal to π, and thus the scheme is not regular. One sees that the non-regularity comes from the form of differencing used for the gradient and divergence terms. Our new scheme is a modification of the central differencing scheme so as to make the scheme regular.

The new scheme we consider will be called the regularized central difference scheme. In this scheme the derivatives of pressure are approximated as

$$\frac{\partial p}{\partial x_k} \approx \delta_{k0} p - \alpha h_k^2 \delta_{k-} \delta_{k+}^2 p$$

and the first derivatives of the velocity in the divergence equation are approximated as

(3.3)
$$\frac{\partial u^k}{\partial x_k} \approx \delta_{k0} u^k - \alpha h_k^2 \delta_{k+} \delta_{k-}^2 u^k \ ,$$

where α is a non-zero constant and δ_{k0}, δ_{k+} and δ_{k-} are the centered, forward, and backward divided differences, respectively. The Laplacian is approximated with the usual five-point scheme. For a cartesian grid in two dimensions the determinant of the symbol is

$$\det \begin{pmatrix} 4h^{-2}(\sin^2 \frac{1}{2} \zeta_1 + \sin^2 \frac{1}{2} \zeta_2) & 0 & d(\zeta_1) \\ 0 & 4h^{-2}(\sin^2 \frac{1}{2} \zeta_1 + \sin^2 \frac{1}{2} \zeta_2) & d(\zeta_2) \\ -\overline{d(\zeta_1)} & -\overline{d(\zeta_2)} & 0 \end{pmatrix}$$

$$= 4h^{-2}(\sin^2 \frac{1}{2} \zeta_1 + \sin^2 \frac{1}{2} \zeta_2)(|d(\zeta_1)|^2 + |d(\zeta_2)|^2)$$

where

$$d(\zeta) = ih^{-1}\sin \zeta - \alpha h^{-1} e^{\frac{1}{2} i\zeta}(2i \sin \frac{1}{2} \zeta)^3$$

$$= 2ih^{-1}\sin \frac{1}{2} \zeta (\cos \frac{1}{2} \zeta + 4\alpha e^{\frac{1}{2} i\zeta}\sin^2 \frac{1}{2} \zeta) \ .$$

Since $d(\zeta)$ is not zero for any value of ζ, when α is non-zero, the scheme is regular. Note that for α equal to one-sixth the approximations (3.2) and (3.3) are third-order accurate.

Since the regularized central difference scheme is a variant of the central difference scheme it is easy to implement with coordinate maps. At those boundary points where the correction term would require points beyond the boundary we use the correction term which interchanges the forward and backward operators. This scheme also requires the use of extrapolation to

compute the pressure values on the boundary. It has been found that third order extrapolation gave quite good results, e.g.

$$(3.4) \qquad p_{0j} = 3p_{1j} - 3p_{2j} + p_{3j}$$

at the boundary $x = 0$ in two dimensions.

A number of first-order accurate schemes for the Stokes and Navier-Stokes equations have been presented e.g. Kzivickii and Ladyzhenskaya (1966) and Temam (1979, p. 48). In this paper we are concerned only with second-order accurate schemes.

4. THE INTEGRABILITY CONDITION.
Each of the schemes for the Stokes equations which have been discussed in the previous section can be written as

$$\text{a)} \quad L_h \vec{u}_h + \vec{G}_h p_h = \vec{f}_h$$

$$(4.1) \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega_h$$

$$\text{b)} \quad \vec{D}_h \cdot \vec{u}_h = g_h$$

with Dirichlet boundary conditions

$$\vec{u}_h = \vec{b}_h \quad \text{on } \partial \Omega_h .$$

The differnce operators L_h, $\vec{G}_h$, and $\vec{D}_h$ are approximations to the differential operators in (1.1). The discrete functions $\vec{f}_h$, g_h, and $\vec{b}_h$ are approximations to f, g and b on the mesh Ω_h, where h is some measure of the fineness of the mesh Ω_h.

Now let us compare the system (4.1) with the system (1.1). First note that if G_h is a consistent approximation to the gradient then the discrete pressure p_h is determined only up to a constant. This means that the system of linear equations (4.1) does not have full column rank. If there are as many equations in (4.1) as there are unknowns, and this is the case for each scheme we've considered, then the system (4.1) does not have full row rank either. This implies that there is a constraint which the data must satisfy to guarantee a solution, in particular, the discrete integrability condition analogous to (1.2) must be satisfied.

There are at least two ways to satisfy the discrete integrability condition. The first method would be to analyze the matrix corresponding to (4.1) and determine the null space of the adjoint matrix. If the data is constrained to be orthogonal to this null space then a solution will exist. This approach is impractical for many situations especially if coordinate changes have been employed since then the matrices are not easy to analyze.

A second approach, which will be adopted here, is to replace (4.1b) by

$$(4.1b') \qquad \vec{D}_h \cdot \vec{u}_h = g_h + \delta_h$$

where δ_h is a constant chosen to guarantee a solution. The value of δ_h

must be determined as part of the solution. As shown in the example: in Section 5 δ_h is at least $O(h^2)$ for the regularized central scheme. We will refer to the equations (4.1a, b', c) as (4.1').

It is interesting to note that for the staggered mesh scheme on a uniform grid one can easily satisfy the discrete integrability condition since the calculus of finite differences mimics the differential calculus very closely, see e.g. Kzivickii and Ladyzhenskaya (1966). Also, Ghia, Hankey, and Hodge (1977) mention being unable to obtain a solution to the discrete Navier-Stokes equations for certain situations. We conjecture that this difficulty was caused by the discrete integrability condition not being satisfied.

There is the possibility that the null space of the discrete operator of (4.1) has dimension greater than one. The regularized central scheme with the third-order extrapolation (3.4) appears to have only a one-dimensional null space. However, for α equal to zero numerical experiments indicate that there are solutions which are effectively null vectors in that they solve (4.1) with f_h and g_h smaller than the norm of the solution by a factor proportional to h or h^2. The dimension of the space of nearly null vectors and null vectors appears to be four for the central differencing scheme. These vectors correspond to the four zeroes of the determinant of the symbol of the difference operator.

These nearly null vectors and null vectors, other than the usual constant pressure null solution, make solving the discrete system very difficult. On the other hand the regular discrete systems can be solved easily by the iterative procedure given in Strikwerda (1982).

5. **COMPUTATIONAL RESULTS.** In this section we present the results of testing the new scheme described in Section 3. In the examples discussed here the discrete Stokes equations were solved using test problems which illustrate various features of the schemes. For each example an exact analytical solution is known and the approximate solutions were compared to the exact solutions to study the accuracy of the method. The value of α, the regularity parameter, was one-sixth in all cases.

For the first test problem the Stokes equations were solved on the unit square with a uniform grid. The exact soltion is

$$u = (2\pi)^{-1} \sin \pi x \cos \pi y$$

(5.1)
$$v = (2\pi)^{-1} \cos \pi x \sin \pi y$$

$$p = \cos \pi x \cos \pi y$$

with $\vec{f} = 0$ and $g = \cos \pi x \cos \pi y$. For this example both the accuracy and symmetry of the solution were checked. The symmetry was checked to study the effect of the nonsymmetric regularizing term on the symmetry of the solution. The symmetry was measured by computing the quantities sym(u) and sym(p) given by

$$\text{sym}(u) = \Big(\sum_{i,j=0}^{N} (u_{ij} + u_{N-i,N-j})^2 \Big)^{1/2} / \|u\|_2$$

(5.2)

$$\text{sym}(p) = \Big(\sum_{i,j=0}^{N} (p_{ij} - p_{N-i,N-j})^2 \Big)^{1/2} / \|p - \bar{p}\|_2$$

for an $(N + 1) \times (N + 1)$ grid. The quantity $\bar{p}$ is the average value of the p_{ij} and the norm is the ℓ^2-norm, e.g.

$$\|u\|_2 = \Big(\sum_{i,j} u_{ij}^2 \Big)^{1/2}.$$

The second test problem demonstrates the ability of the scheme to produce second-order accurate solutions on a non-rectangular region. The exact solution is

$$u = \xi^2 + \eta^2$$

$$v = -2\xi\eta + \eta^2$$

$$p = 4\xi + 2\eta$$

on the region Ω which is the image of the unit square under the mapping

$$\xi = x \cosh(y)$$

$$\eta = y - x^2$$

for (x,y) in the unit square, i.e. $0 \leqslant x, y \leqslant 1$. Thus the equations being solved on the unit square were

$$x_\xi (x_\xi u_x)_x + x_\xi (y_\xi u_y)_x + y_\xi (x_\xi u_x)_y + y_\xi (y_\xi u_y)_y$$

$$+ x_\eta (x_\eta u_x)_x + x_\eta (y_\eta u_y)_x + y_\eta (x_\eta u_x)_y + y_\eta (y_\eta u_y)_y - x_\xi p_x - y_\xi p_y = 0$$

for the first equation, with the second being similar, and

$$x_\xi u_x + y_\xi u_y + x_\eta v_x + y_\eta v_y = 0$$

for the third equation. The regularizing terms were added only to the terms corresponding to p_x in the first equation, p_y in the second, and u_x and v_y in the third.

In the third test problem the Stokes equations were solved on a disc using polar coordinates with uneven grid spacing in both the radial and angular direction. The exact solution is

$$u = r^3 \sin 2\theta$$

$$v = 2r^3 \cos 2\theta$$

$$p = 6r^2 \sin 2\theta$$

with $\vec{f}$ and g being zero. The uneven grid was given by

$$r_i = .75 \, \rho_i + .25 \, \rho_i^2$$

$$\theta_j = \varphi_j - .25 \sin \varphi_j$$

where ρ_i and φ_j were evenly spaced in the interval $[0,1]$ and $[0,2\pi]$ respectively. This uneven spacing was chosen merely to show the versatility of the scheme and is not intended to give a better resolution of the solution.

For completeness we give the Stokes equations in polar coordinates

$$r^{-1}(ru_r)_r + r^{-2}u_{\theta\theta} - r^{-2}u - 2r^{-2}v_\theta - p_r = 0$$

(5.5)
$$r^{-1}(rv_r)_r + r^{-2}v_{\theta\theta} - r^{-2}v + 2r^{-2}u_\theta - r^{-1}p_\theta = 0$$

$$r^{-1}(ru)_r + r^{-1}v_\theta = 0 .$$

The difference formulas used in the numerical experiments were all second order accurate. As an example of the formulas, the term $r^{-1}(ru_r)_r$ was differenced as

$$\left(\frac{r_{i+1} + r_i}{r_{i+1} - r_i}\right)(u_{i+1,j} - u_{i-j}) - \left(\frac{r_i + r_{i-1}}{r_i - r_{i-1}}\right)(u_{i,j} - u_{i-1,j}) / \frac{1}{2}\left(r_{i+1}^2 - r_{i-1}^2\right) .$$

The results of the numerical experiments are shown in the following tables. Each table lists the errors incurred for grids with $N + 1$ points on a side for values of N of 20, 30, 40 and 60. Tables I, II and III list the relative errors for test problems 1, 2 and 3, respectively, and Table I also shows the symmetry errors for problem 1. The relative errors are measured in the ℓ^2-norm i.e.

$$\text{err}(u) = \left(\sum (u_{ij} - u(x_i,y_i))^2\right)^{1/2} / \|u\|_2 .$$

Also shown is the value of δ_h which is described in Section 4. Table IV displays the behavior of the error as the grid resolution is increased. The numbers shown are values of

$$-\frac{\log(\text{err}_1/\text{err}_2)}{\log(N_1/N_2)}$$

TABLE I

N	err(u)	err(p)	δ_h	sym(u)	sym(p)
20	.35(-3)	.17(-2)	-.44(-5)	.68(-3)	.13(-2)
30	.11(-3)	.86(-3)	-.89(-6)	.22(-3)	.37(-3)
40	.41(-4)	.51(-3)	-.53(-6)	.82(-4)	.15(-3)
60	.19(-4)	.23(-3)	-.50(-7)	.37(-4)	.52(-4)

Errors for test problem 1 for grids with $N + 1$ points on a side for four values of N. The numbers in parenthesis are the decimal exponents i.e. $-35(-3) = .35 \times 10^{-3}$.

TABLE II

N	err(u)	err(p)	δ_h
20	.10(-3)	.21(-2)	-.24(-3)
30	.45(-4)	.92(-3)	-.12(-3)
40	.25(-4)	.48(-3)	-.74(-4)
60	.11(-4)	.22(-3)	-.35(-4)

Errors for test problem 2.

Table III

N	err(u)	err(p)	δ_h
20	.75(-1)	.93(-1)	-.33(-2)
30	.33(-1)	.34(-1)	-.53(-3)
40	.19(-1)	.18(-1)	-.15(-3)
60	.83(-2)	.75(-2)	-.27(-4)

Errors for test problem 3.

TABLE IV

N_1/N_2		1	2	3
	u	2.8	2.0	2.0
30/20	p	1.7	2.0	2.5
	δ_h	4.0	1.7	4.5
	u	3.4	2.0	1.9
40/30	p	1.8	2.3	2.2
	δ_h	1.9	1.7	4.4
	u	3.1	2.0	2.0
40/20	p	1.7	2.1	2.4
	δ_h	3.1	1.7	4.5
	u	2.5	2.0	2.0
60/30	p	1.9	2.1	2.2
	δ_h	4.2	1.8	4.3
	u	1.9	2.0	2.0
60/40	p	2.0	1.9	2.2
	δ_h	5.8	1.8	4.2

Computed order of accuracy for u, p, and δ_h for the test problems.

where err_1 and err_2 are the errors for grids of $N_1 + 1$ and $N_2 + 1$ points on a side, respectively. This value should be approximately 2.0 for a second-order scheme. The error reductions are shown for u, p and δ_h. The other velocity component had a similar error behavior in all the examples. All of the solutions were computed by the iterative method given in Strikwerda (1982).

That some of the errors were better than second-order accurate for test problems 1 and 3 can be attributed to the third-order accurate difference formulas used for the gradient and divergence terms. One might expect that some of the errors would behave as third-order errors for some value of N_1 and N_2. However, since the discrete Laplacian is second-order accurate, for N large enough the total scheme should be second-order accurate. It is not clear why δ_h should behave as a fourth-order error as seen in test problem 3 and for some values of N_1 and N_2 in test problem 1. Test problem 2 was no better than second-order accurate since the gradient and divergence were only second-order accurate. The third-order differences were only used on those terms which were necessary to achieve regularity of the scheme. The results show conclusively that the scheme has overall second-order accuracy.

6. CONCLUSION. In this paper we have examined several finite difference methods for the steady Stokes and incompressible Navier-Stokes equations in primitive variables. We have shown that the regularized centered difference scheme is second-order accurate and useful with non-rectangular regions. Although the numerical experiments were done using the Stokes equations, for which exact solutions were available, we believe the regularized central scheme is equally useful with the incompressible Navier-Stokes equations at moderate Reynolds number.

REFERENCES

S. Agmon, A. Douglis, L. Nirenberg (1964). Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II, Comm. Pure Appl. Math. 17, pp. 35-92.

A. Brandt, N. Dinar (1979). Multi-grid solutions to elliptic flow problems, Proc. of Conference on Numerical Solutions of Part. Diff. Equations, Math. Res. Center, Madison, WI, October 1978.

K. Bube, J. Strikwerda (1980). Interior regularity estimates for elliptic systems of difference equations, ICASE Report 8-28, SIAM J. Num. Anal. to appear.

A. Chorin (1967). A numerical method for solving incompressible viscous flow problems, J. Comp. Phys., 2, pp. 12-26.

A. Chorin (1968). Numerical solution of the Navier-Stokes equations, Math. Comput., 22, pp. 745-762.

A. Douglis, L. Nirenberg (1955). Interior estimates for elliptic systems of partial differential equations, Comm. Pure Appl. Math., 8, pp. 503-538.

L. Frank (1968). Difference operators in convolutions, Soviet Math. Dokl., 9, pp. 831-834.

K. N. Ghia, W. L. Hankey, J. K. Hodge (1977). Study of incompressible Navier-Stokes equations in primitive variables using implicit numerical technique, AIAA paper 77-648.

F. H. Harlow and J. E. Welch (1964). Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface, Physics of Fluids, 8, pp. 2181-2189.

A. Kzivickii, O. A. Ladyzhenskaya (1966). The method of nets for the non-stationary Navier-Stokes equations, Proc. of the Steklov Inst., 92, pp. 105-112.

O. A. Ladyzhenskaya (1963). The Mathematical Theory of Viscous Incompressible Flows, Translated by R. A. Silverman, Gordan and Breach, New York.

S. V. Patankar, D. B. Spalding (1972). A calculation procedure for heat, mass, and momentum transfer in three dimensional parabolic flows, Internat. J. Heat and Mass Trans., 15, pp. 1787-1806.

G. D. Raithby, G. E. Schneider (1979). Numerical solution of problems in incompressible fluid flow: Treatment of the velocity-pressure coupling, Num. Heat Transf., 2, 417-440.

P. Roache (1972). Computational Fluid Dynamics, Hermosa Publ., Albuquerque, NM.

S. Rubin, J. Harris, ed. (1975). Numerical studies of incompressible viscous flow in a driven cavity, NASA SP-378.

J. C. Strikwerda (1982). An iterative method for solving the Stokes equations, to appear.

T. D. Taylor, E. Ndefo (1970). Computation of viscous flow in a channel by the method of splitting, Proc. Second Int. Conf. Numer. Methods Fluid Dynamics, p. 356-364.

R. Temam (1979). Navier-Stokes Equations, North Holland Publ. Co., Amsterdam.

N. N. Yanenko (1971). The Method of Fractional Steps; the Solution of Problems of Mathematical Physics in Several Variables. Translated by M. Holt, Springer-Verlag, Berlin.

JCS/ed

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2361	2. GOVT ACCESSION NO. AD-A116.177	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) FINITE DIFFERENCE METHODS FOR THE STOKES AND NAVIER-STOKES EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) John C. Strikwerda		8. CONTRACT OR GRANT NUMBER(s) NASI-16394 DAAG29-80-C-0041 NASI-15810
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Computer Science
11. CONTROLLING OFFICE NAME AND ADDRESS (See Item 18 below)		12. REPORT DATE April 1982
		13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office	ICASE
P. O. Box 12211	NASA Langley Research Center
Research Triangle Park	Hampton, VA 23665
North Carolina 27709	

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Finite Differences, Navier-Stokes Equations, Stokes Equations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper presents a new finite difference scheme for the Stokes equations and incompressible Navier-Stokes equations for low Reynold's number. The scheme uses the primitive variable formulation of the equations and is applicable with non-uniform grids and non-rectangular geometries. Several other methods of solving the Navier-Stokes equations are also examined

(cont.)

DD FORM _{1 JAN 73} 1473   EDITION OF 1 NOV 65 IS OBSOLETE

ABSTRACT (cont.)

in this paper and some of their strengths and weaknesses are described.
Computational results using the new scheme are presented for the Stokes
equations for a region with curved boundaries and for a disc with polar
coordinates. The results show the method to be second-order accurate.

DATE

ILME

—8